Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. XIV. AUGUST-SEPTEMBER, 1907. NOS. 8-9.

AN EXAMPLE OF THE INDICATRIX IN THE CALCULUS OF VARIATIONS.*

By ARNOLD DRESDEN, The University of Chicago.

(Continued.)

§5. *Case I:* $0<r<a$.

From our previous work, we obtain

$$\mathrm{F}(r,\psi)>0,\quad \mathrm{F}_1(r,\psi)>0,\quad 0\leqq\psi\leqq 2\pi.$$

We conclude from this, by theorems *A* and *B* (see page 121), that the Indicatrix for any point in this region is a curve, which is closed around the origin G_1 and which has constantly positive curvature (see Fig. 5, curve *A*). It follows, that all the tangents lie entirely outside the curve, so that all points $\overline{Q}$ lie on the same side of the tangent at *any* point *Q*, as the origin *G*. Since, moreover, $\mathrm{F}>0$, we have by theorem *C* (see page 122), that

$$E(x,\, y;\, \cos\theta,\, \sin\theta;\, \cos\overline{\theta},\, \sin\overline{\theta})>0,$$

for all pairs $(\theta, \overline{\theta})$, $0\leqq\theta\leqq 2\pi$, $0\leqq\overline{\theta}\leqq 2\pi$, except when $\theta=\overline{\theta}$.

But this, by Nos. II and IV of our general theorems (see page 120) expresses the following result:

II. *Every extremal-element*† (x, y, ψ) *in the region I is an element of a hyperstrong minimum.*

This result could have been obtained more simply, by observing that since $F_1>0$, for all values of ψ between 0 and 2π, condition (IV′) is satisfied.

It follows further, that the curve does not admit double tangents, so that *no point in the region I can be the corner of a discontinuous solution,* which is evident by theorem *D* (page 123).

---

*For the first part of this paper, see the previous number of this journal.

†To say, that (x, y, ψ) is an extremal element is a short way of expressing, that it is possible to draw through the point (x, y) an extremal in a direction, which makes an angle ψ with the radius-vector at (x, y), and consequently an angle θ with the positive X-axis (compare p. 124, equation (9) and Fig. 2).

§ 6. *Case II:* $a<r<\sqrt{3}$.

We have here:

$$
\begin{aligned}
&F(r,\psi)>0, \quad 0\leqq\psi\leqq 2\pi,\\
&F_1(r,\psi)>0, \quad \text{for } 0\leqq\psi<\alpha,\\
&F_1(r,\psi)=0, \quad \text{for } \psi=\alpha,\\
&F_1(r,\psi)<0, \quad \text{for } \alpha<\psi\leqq\tfrac{1}{2}\pi,
\end{aligned}
$$

where α is defined by

$$\alpha\equiv\sin^{-1}\left(\frac{a}{r}\right), \quad 0\leqq\alpha\leqq\frac{\pi}{2}. \tag{12}$$

Applying theorems *A* and *B* to these results, we find, that the Indicatrix for a point in this region is a curve, closed around the origin, which changes curvature at $\psi=\alpha$, $\pi-\alpha$, $\pi+\alpha$ and $2\pi-\alpha$, having positive curvature in the intervals $(2\pi-\alpha, \alpha)$ and $(\pi-\alpha, \pi+\alpha)$, and negative curvature in the intervals $(\alpha, \pi-\alpha)$ and $(\pi+\alpha, 2\pi-\alpha)$.

The form of equation (11) shows us further, that the curve is symmetric with respect to the *X*-axis and with respect the *Y*-axis. To determine its character more accurately, we denote by $\bar{\psi}$ the angle between tangent and radius vector and obtain:

$$\tan.\bar{\psi}=\frac{\rho}{\rho'}=\frac{F}{F'}=\frac{4(1+r^2\sin^2\psi)-(r^2+1)\sqrt{(1+r^2\sin^2\psi)}}{4r^2\sin\psi\cos\psi} \tag{13}.$$

Since $4(1+r^2\sin^2\psi)-(r^2+1)\sqrt{(1+r^2\sin^2\psi)}=4(r^2+1)\sqrt{(1+r^2\sin^2\psi)}\cdot F$,

and moreover for this case $F>0$, it follows, that $\tan\bar{\psi}\neq 0$.

Further, $\tan\bar{\psi}=\infty$, for $\psi=0$, $\psi=\frac{1}{2}\pi$; *i. e.*, the curve crosses at right angles at $\psi=0$ and at $\psi=\frac{1}{2}\pi$ (see Fig. 5, curve *B*). It is finally of importance, to determine the value of ψ, different from $\frac{1}{2}\pi$, for which the curve has a horizontal tangent. Denoting that value by β and transforming to rectangular coordinates:

$$\left.\begin{aligned}\xi&=\rho\cos\psi\\ \eta&=\rho\sin\psi\end{aligned}\right\}, \tag{14}$$

we obtain:

$$\eta'=\rho'\sin\psi+\rho\cos\psi=\frac{\cos\psi[4-(1+r^2)\sqrt{(1+r^2\sin^2\psi)}]}{4(1+r^2)\sqrt{(1+r^2\sin^2\psi)}F^2}.$$

Consequently,

$$\sin^2\beta=\frac{(r^2+5)\,(3-r^2)}{r^2(1+r^2)^2}. \tag{15}$$

Restricting ourselves in our further discussion of the curve to one quadrant, we find:

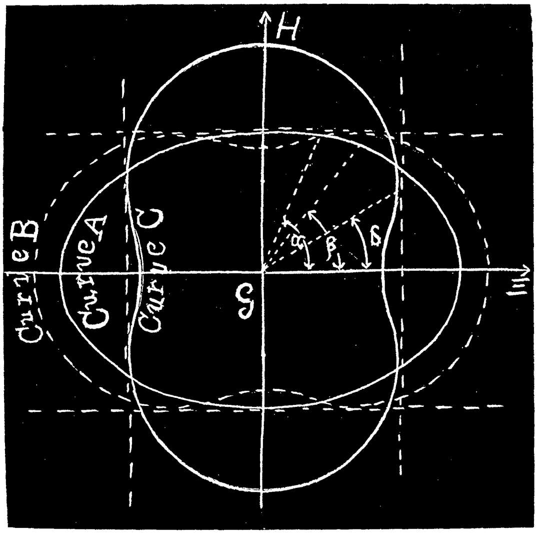

Fig. 5.

For $0 \leqq \psi < \beta$; every tangent lies outside the curve,—it has positive curvature.

For $\psi = \beta$, the curve has a double tangent, touching at β and $\pi - \beta$.

For $\beta < \psi < \alpha$; every tangent intersects the curve,—it has positive curvature.

For $\alpha < \psi < \frac{1}{2}\pi$; every tangent intersects the curve,—it has negative curvature.

By applying rows No. II and IV of our general theorems, we obtain:

$0 \leqq r \sin\psi \leqq r \sin\beta$	$E(x, y; \cos\theta, \sin\theta; \cos\bar{\theta}, \sin\bar{\theta}) < 0$ for $0 \leqq \bar{\theta} \leqq 2\pi$, except $\theta = \bar{\theta}$. $F_1 > 0$.	(x, y, ψ) is an element of strong minimum.
$r \sin\psi = r \sin\beta$	$F_{x'}(x, y, \cos\psi, \sin\psi) = F_{x'}(x, y, \cos(r-\psi), \sin(\pi-\psi))$. $F_{y'}(x, y, \cos\psi, \sin x) = F_{y'}(x, y, \cos(r-\psi), \sin(r-\psi))$.	(x, y, ψ) and $(x, y, r-\psi)$ are elements of discontinuous solutions.
$r \sin\beta < r \sin\psi \leqq a$	$E(x, y; \cos\theta, \sin\theta; \cos\bar{\theta}, \sin\bar{\theta})$ is of changing sign for $0 \leqq \bar{\theta} \leqq 2\pi$. $F_1 > 0$.	(x, y, ψ) is an element of weak minimum.
$a < r \sin\psi < r$	E as above. $F_1 < 0$.	(x, y, ψ) is an element of weak maximum.

Denoting row $r \sin\psi$ by d, and rembering that it represents the distance from the origin to a straight line, we observe that if $r \sin\psi$ has a certain value for one element, it will have the same value along the entire line, to which the element belongs.

We find then, that straight line segments, which lie outside C_a are *weak maxima*, which agrees with a previous result (see I on page 125).

The elements, which belong to lines intersecting C_a, belong to weak minima, if

$$d > \frac{\sqrt{[(r^2+5)(3-r^2)]}}{1+r^2},$$

to strong minima, if

$$d < \frac{\sqrt{[(r^2+5)(3-r^2)]}}{1+r^2}.$$

Solving the equation

$$d = \frac{\sqrt{[(r^2+5)(3-r^2)]}}{1+r^2}$$

for r, we obtain

$$r = \sqrt{\sqrt{\frac{16}{d^2+1}}-1}, \tag{16}$$

which enables us to determine on every line, intersecting C_a, a point which separates the segments which are strong minima, from those which are weak minima. Constructing this point for all extremals which are parallel to the X-axis, we obtain the curve, denoted by C_r in Figs. 4 and 6. For other extremals, an analogous curve can be obtained by revolving C_r until the line PP' becomes parallel to the given extremal.*

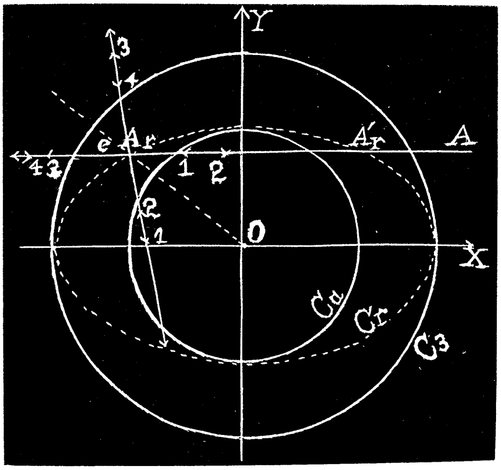

Fig. 6.†

We see further, that since (x, y, φ) is an element of discontinuous solution, every point in this region is a corner of four discontinuous solutions, the branches being in the directions β, $\pi-\beta$, $\pi+\beta$, and $2\pi-\beta$, as indicated in Fig. 6, where 1 and 2, as well as 3 and 4 must be considered as distinct (compare A. Kneser, *Lehrbuch der Variationsrechnung*, p. 9).

From the definition of C_r follows finally, that the intersection of C_r with a line parallel to X-axis (or of an analogous curve with any other line) determine upon that line two points, which are each corners of four discontinuous solutions, of which the given line furnishes one of the branches. The other branches are obtained by reflecting the line on the radius vector of the point. In Fig. 6, where A is the given line, A_r and A'_r are the points referred to, while the other branches for the former point are formed by BB'.

---

*For practical purposes, one would make a cardboard sample of C_r and put it in any required position to determine the points corresponding to A_r and A'_r (Fig. 6).

†ERRATUM.—In the diagram, 4 and 3 on the oblique line through A_r should be interchanged. ED. F.

We have now the following result:

III. *A straight line not intersecting C_a, is a weak maximum within C_3. A straight line intersecting C_a is a strong minimum between C_a and C_r, a weak minimum between C_r and C_3.*

C_r determines upon every line intersecting C_a a corner of a discontinuous solution.

Every point between C_a and C_3 can be a corner of a discontinuous solution.

§ 7. *Case III.* $\sqrt{3}<r<\sqrt{15}$.

In this case we find:

$$\begin{aligned} &F(r,\ \psi)<0, \text{ for } 0\leqq \psi<\gamma, \\ &F(r,\ \psi)=0, \text{ for } \psi=\gamma, \\ &F(r,\ \psi)>0, \text{ for } \gamma<\psi\leqq \tfrac{1}{2}\pi, \end{aligned}$$

where γ is defined by

$$\frac{\sqrt{(1+r^2\sin^2\gamma)}}{1+r^2}-\tfrac{1}{4}=0,\ 0\leqq\gamma\leqq\tfrac{1}{2}\pi;\ i.\ e.,\ \sin^2\gamma=\frac{(r^2+5)\ (r^2-3)}{16r^2} \tag{17}$$

$F_1(r,\ \psi)$ as in § 6.

Application of theorems A and B shows, that in this case, the Indicatrix has negative curvature in the interval $0\leqq\psi<\gamma$, and has a discontinuity at $\psi=\gamma$. It has further positive curvature from $\psi=\gamma$ to $\psi=a$, from where on, it continues with negative curvature until $\psi=\frac{1}{2}\pi+a$.* The further course of the curve is determined by its symmetry properties.

From equation (13) follows again, that the curve crosses at right angles at $\psi=0$, and at $\psi=\frac{1}{2}\pi$; further $\tan\bar{\psi}=0$, for $\psi=\beta$, *i. e.*, the curve is tangent to a line parallel to the radius vector at its point of discontinuity. To determine the asymptote, we compare the curve at $\psi=\beta$ and the straight line $\psi=\beta$. Denoting the rectilinear co-ordinates of the curve by ξ and η, and those of the straight line by $\bar{\xi}$ and $\bar{\eta}$, we find for a given $\xi=\bar{\xi}$:

$$\eta=\rho\sin\psi=\frac{\sin\psi}{F},\quad \bar{\eta}=\tan\beta.\bar{\xi}=\tan\beta.\xi=\frac{\tan\beta\cos\psi}{F}.$$

---

*This discussion is made under the assumption that $\gamma<a$; if $a<\gamma$, the curve is of a somewhat different nature. This does not affect the conclusions derived in this consideration however, for which reason the cases are not distinguished. Comparing equations (12) and (17), we find that

$$a\gtreqless\gamma,\ \text{for}\ r\lesseqgtr\sqrt{(4^{\frac{4}{3}}-1)}.$$

Consequently:

$$\eta-\bar{\eta}=\frac{\sin(\psi-\beta)}{\cos\beta\,\mathrm{F}},$$

and $$\underset{\psi=\beta}{\mathrm{L}}(\eta-\bar{\eta})=\frac{16(r^2+1)}{(15-r^2)\sqrt{[(r^2+5)(r^2-3)]}}, \tag{18}$$

which is real and positive for all points between C_3 and C_{15}. It follows, that the asymptote is parallel to the line $\psi=\beta$, and intersects the Y-axis at the point:

$$\eta=\frac{16(r^2+1)}{(15-r^2)\sqrt{[(r^2+5)(r^2-3)]}}.$$

We conclude now immediately, that every tangent intersects the curve, and that no double tangents are possible, from which by theorems *C, D* and IV, V, and by the reasoning given in § 6 for the connection between extremal elements and extremals, we derive the following results:

IV. *A straight line, not intersecting C_a is a weak maximum between C_3 and C_{15}.*

A straight line, intersecting C_a is a weak minimum between C_3 and C_{15}.

No point in the region between C_3 and C_{15} can be a corner of a discontinuous solution.

§ 8. *Case IV.* $\sqrt{15}<r$.

We have here finally:

$\mathrm{F}(r,\psi)<0,\ 0\leqq\psi\leqq2\pi$.

$\mathrm{F}_1(r,\psi)$ as in § 6.

We find here, by application of theorems *A* and *B*, that the Indicatrix is closed around the origin, has negative curvature for $0\leqq\psi<\alpha$, and positive curvature for $\alpha<\psi\leqq\frac{1}{2}\pi$, again having double symmetry. As in case II, the curve crosses at right angles for $\psi=0$ and $\psi=\frac{1}{2}\pi$, and we want to determine the value of ψ, different from 0 for which it has a vertical tangent. Denoting that value by δ, we find by transforming to rectangular coordinates (14), by a method similar to the one used for the determination of β:

$$\sin^2\delta=\frac{15}{r^2}, \tag{19}$$

which furnishes a real value of δ for any point outside C_{15} (see Fig. 5, curve *C*).

The case bears very much likeness to the one discussed in § 6, for which reason, we conclude briefly, using theorems *C, D,* IV, and V:

$0 \leqq r \sin.\varphi \leqq a$	(x, y, φ) is an element of weak minimum.
$a < r \sin.\varphi < \sqrt{15}$	(x, y, φ) is an element of weak maximum.
$r \sin.\varphi = \sqrt{15}$	(x, y, φ) and $(x, y, 2\pi - \varphi)$ are elements of discontinuous solutions.
$\sqrt{15} < r \sin.\varphi$	(x, y, φ) is an element of strong maximum.

We notice, that the branches of the discontinuous solutions are determined by the tangents to C_{15}, so that, although every point of the region is a corner of four discontinuous solutions (see Fig. 7), we can not determine on *every* line a corner of a discontinuous solution of which the given line is a branch, as could be done in case II, whereas on some lines we can determine an infinitude of corners, which was impossible in II.

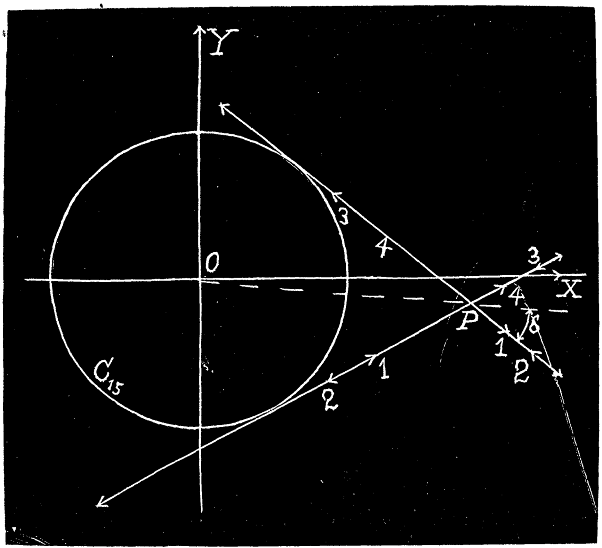

Fig. 7.[*]

We have here the following result:

V. A straight line intersecting C_a is a weak minimum outside C_{15}. A straight line, not intersecting C_a, but intersecting C_{15}, is a weak maximum outside C_{15}. A straight line, lying entirely outside C_{15}, is a strong maximum.

Every point outside C_{15} is a corner of four discontinuous solutions, which can be determined by drawiag the tangents to C_{15}.

§ 9. *Limiting cases and final results.*

We consider in conclusion:

---

[*] ERRATUM.—δ should be between the dotted line and the lower oblique line through P.

(a) $r=a$. We have then:

$F(r, \varphi)>0,\ 0\leqq\varphi\leqq 2\pi$.

$F_1(r, \varphi)>0,\ 0\leqq\varphi\leqq 2\pi$, excepting $\varphi=\frac{1}{2}\pi$ or $\frac{3}{2}\pi$.

The Indicatrix is of the same nature as for case I. Consequently, any extremal-element on C_a is an element of a *strong minimum*, while no discontinuous solutions can have a corner on C_a.

(b) $r=\sqrt{3}$.

$F(r, \varphi)=0$ for $\varphi=0$.

$F(r, \varphi)>0$ for $0<\varphi\leqq\frac{1}{2}\pi$.

$F_1(r, \varphi)$ as in § 6.

The Indicatrix is of the nature described in § 7, and we find from equation (18):

$$\underset{\varphi\doteq\beta}{\mathrm{L}}(\eta-\bar{\eta})=\infty,$$

from which follows, that the curve has a parabolic character, so that we conclude: Any extremal-element in C_3 is an element of a *weak minimum*, if $\varphi<\alpha$; it is an element of a *weak maximum*, if $\varphi>\alpha$.

(c) $r=\sqrt{15}$.

$F(r, \varphi)>0$, for $0\leqq\varphi<\frac{1}{2}\pi$.

$F(r, \varphi)=0$, for $\varphi=\frac{1}{2}\pi$.

$F_1(r, \varphi)$ as in § 6.

We find again from equation (19):

$$\underset{\varphi=\beta}{\mathrm{L}}(\eta-\bar{\eta})=\infty,$$

from which we derive the same results as in sub (b).

We combine now the results I, II, III, IV, and V, obtained §§ 3, 5, 6, 7, 8 for lines parallel to the X-axis (see Fig. 4):

1. *D is a strong maximum along its entire length and allows four discontinuous solutions at every point.*

2. *C is a weak maximum along its entire length; every point outside of the interval $C'_{15}C_{15}$ is a corner of four discontinuous solutions.*

3. *B is a weak maximum along its entire length; every point on the segment B'_3B_3 is a corner of four discontinuous solutions.*

4. *A is a weak minimum outside the segment* A'_rA_r; it is a strong minimum from A_a up to A_r and from A'_a up to A'_r; it is a hyperstrong minimum on the segment A'_aA_a; all points outside the interval $A'_{15}A_{15}$ and all points on the segments $A'_3A'_a$ and A_3A_a are the corners of four discontinuous solutions; A_r in particular is the corner of the discontinuous solutions, of which one branch lies along A.*

---

*The segment AB excludes the points A and B; the interval AB includes the points A and B.